

NON-SEPARABLE TREE-LIKE BANACH SPACES AND ROSENTHAL'S ℓ_1 -THEOREM

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ABSTRACT. We introduce and investigate a class of non-separable tree-like Banach spaces. As a consequence, we prove that we can not achieve a satisfactory extension of Rosenthal's ℓ_1 -theorem to spaces of the type $\ell_1(\kappa)$, for κ an uncountable cardinal.

1. INTRODUCTION

Rosenthal's ℓ_1 -theorem [8] is one of the most remarkable results in Banach space geometry. It provides a fundamental criterion for the embedding of ℓ_1 into Banach spaces.

Theorem 1.1 (Rosenthal's ℓ_1 -theorem). *Let (x_n) be a bounded sequence in the Banach space X and suppose that (x_n) has no weakly Cauchy subsequence. Then (x_n) contains a subsequence equivalent to the usual ℓ_1 -basis.*

A satisfactory extension of Theorem 1.1 to spaces of the type $\ell_1(\kappa)$, for κ an uncountable cardinal, would be desirable, since it would provide a useful criterion for the embedding of $\ell_1(\kappa)$ into Banach spaces. Naturally, therefore, R. G. Haydon [6] posed the following problem: Let κ be an uncountable cardinal. Suppose that X is a Banach space, A is a bounded subset of X whose cardinality is equal to κ and such that A does not contain any weakly Cauchy sequence. Can we deduce that A has a subset equivalent to the usual $\ell_1(\kappa)$ -basis?

Before the question was posed, Haydon [5] had already presented a counterexample for the case where the cardinal κ is equal to ω_1 . A completely different counterexample for the case of ω_1 had also been obtained by J. Hagler [3]. Finally, the complete solution to the aforementioned problem was given by C. Gryllakis [2] who proved that the answer is always negative with only one exception, namely when both κ and $cf(\kappa)$ are strong limit cardinals.

In this paper, we first introduce for any infinite cardinal κ a tree-like Banach space X_κ . Our construction is motivated by the well-known James Tree space (JT) [7] and Hagler Tree space (HT) [3]. We also study in detail various properties of the space X_κ and we mostly focus on a family of continuous functionals defined on X_κ . As a consequence of our investigation we give a very simple answer to Haydon's problem.

Closing this introductory section, we recall some definitions for the sake of completeness. A sequence $(x_n)_{n \in \mathbb{N}}$ in a Banach space X is *weakly Cauchy* if the scalar sequence $(f(x_n))_{n \in \mathbb{N}}$ converges for every f in X^* . A subset $A \subset X$ with cardinality

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κ is *equivalent to the usual $\ell_1(\kappa)$ -basis* if there are constants $C_1, C_2 > 0$ such that $C_1 \sum_{i=1}^n |a_i| \leq \|\sum_{i=1}^n a_i x_i\| \leq C_2 \sum_{i=1}^n |a_i|$, for any $n \in \mathbb{N}$, any $x_1, x_2, \dots, x_n \in A$ and any scalars a_1, \dots, a_n .

Finally, we should mention that this is not the first time non-separable tree-like Banach spaces have been defined (e.g. see [1] and [4]).

2. THE BASIC CONSTRUCTION

Suppose that κ is an infinite cardinal. Then we set

$$\begin{aligned} \Gamma &= \{0, 1\}^\kappa = \left\{ a : \{\xi < \kappa\} \rightarrow \{0, 1\} \right\} = \left\{ (a_\xi)_{\xi < \kappa} \mid a_\xi = 0 \text{ or } 1 \right\} \\ \mathcal{D} &= \{0, 1\}^{<\kappa} = \bigcup \left\{ \{0, 1\}^\eta \mid \text{Ord}(\eta), \eta < \kappa \right\} \\ &= \left\{ (a_\xi)_{\xi < \eta} \mid \eta \text{ is an ordinal, } \eta < \kappa, a_\xi = 0 \text{ or } 1 \right\}. \end{aligned}$$

The set \mathcal{D} is called the *(standard) tree*. The elements $s \in \mathcal{D}$ are called *nodes*. The elements of the set $\Gamma = \{0, 1\}^\kappa$ are called *branches*.

If s is a node and $s \in \{0, 1\}^\eta$, we say that s is on the η -th level of \mathcal{D} . We denote the level of s by $\text{lev}(s)$. The *initial segment partial ordering* on \mathcal{D} , denoted by \leq , is defined as follows: if $s = (a_\xi)_{\xi < \eta_1}$ and $s' = (b_\xi)_{\xi < \eta_2}$ belong to \mathcal{D} then $s \leq s'$ if and only if $\eta_1 \leq \eta_2$ and $a_\xi = b_\xi$ for any $\xi < \eta_1$. We also write $s < s'$ if $s \leq s'$ and $s \neq s'$. By $s \perp s'$ we mean that s, s' are *incomparable*, that is neither $s \leq s'$ nor $s' \leq s$. If $s \leq s'$ we say s' is a *follower* of s . Further, the nodes $s \cup \{0\}$ and $s \cup \{1\}$ are called the *successors* of s , that is we reserve the word successor as meaning immediate follower. However, we observe that a node does not need to have an *immediate predecessor*.

A subset T of \mathcal{D} is called a *subtree* if it is order isomorphic to $\{0, 1\}^{<\lambda}$ for some cardinal $\lambda \leq \kappa$. In this paper, we only use countable subtrees of \mathcal{D} , that is subtrees order isomorphic to $\{0, 1\}^{<\aleph_0}$. In the case T is countable, we enumerate its elements as $T = \{t_1, t_2, t_3, \dots\}$ where t_1 is the minimum element of T and for each $m \in \mathbb{N}$, t_{2m}, t_{2m+1} are the successors (on the tree T) of t_m .

A linearly ordered subset \mathcal{I} of \mathcal{D} is called a *segment* if for every $s < t < s'$, t is contained in \mathcal{I} provided that s, s' belong to \mathcal{I} . Consider now a non-empty segment \mathcal{I} . Let η_1 be the least ordinal such that there exists a node $s \in \mathcal{D}$ with $\text{lev}(s) = \eta_1$ and $s \in \mathcal{I}$. Suppose further that there are an ordinal η and a node s' on the η -th level so that $s \leq s'$ for every $s \in \mathcal{I}$. Let η_2 be the least ordinal satisfying this property. Then we say that \mathcal{I} is an η_1 - η_2 segment. A segment is called *initial* if $\eta_1 = 0$, that is $\emptyset \in \mathcal{I}$.

We next define admissible families of segments in the sense of Hagler [3]. Suppose that $\{\mathcal{I}_j\}_{j=1}^r$ is a finite family of segments. This family is called *admissible* if the following conditions are satisfied:

- (1) there exist ordinals $\eta_1 < \eta_2$ such that \mathcal{I}_j is an η_1 - η_2 segment for each $j = 1, \dots, r$;
- (2) $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ provided that $i \neq j$.

Consider now the vector space $c_{00}(\mathcal{D})$ of finitely supported functions $x : \mathcal{D} \rightarrow \mathbb{R}$. For any segment \mathcal{I} of \mathcal{D} , we set $\mathcal{I}^* : c_{00}(\mathcal{D}) \rightarrow \mathbb{R}$ with $\mathcal{I}^*(x) = \sum_{s \in \mathcal{I}} x(s)$. Then,

for any $x \in c_{00}(\mathcal{D})$, we define the norm

$$\|x\| = \sup \left[\sum_{j=1}^r |\mathcal{I}_j^*(x)|^2 \right]^{1/2}$$

where the supremum is taken over all finite, admissible families $\{\mathcal{I}_j\}_{j=1}^r$ of segments. The space X_κ is the completion of the normed space $(c_{00}(\mathcal{D}), \|\cdot\|)$ we have just defined.

For every node $s \in \mathcal{D}$, we define $e_s : \mathcal{D} \rightarrow \mathbb{R}$ with $e_s(t) = 1$ if $t = s$ and $e_s(t) = 0$ otherwise. Clearly, $\|e_s\| = 1$ for any $s \in \mathcal{D}$.

We come now to the final definition. Suppose that $\{s_i \mid i \in I\}$ is a family of nodes of the tree \mathcal{D} . This family is called *strongly incomparable* (see [3]) if the following hold:

- (1) $s_i \perp s_j$ provided that $i \neq j$;
- (2) if $\{S_1, \dots, S_r\}$ is any admissible family of segments, then at most two nodes of the s_i 's, $i \in I$, are contained in $S_1 \cup \dots \cup S_r$.

There is a standard way for constructing strongly incomparable families of nodes. Suppose that $(s_\xi)_{\xi < \eta}$ is a set of nodes, where $\eta < \kappa$, such that $s_0 < s_1 < \dots$. For any ordinal $\xi < \eta$, let t_ξ be the successor of s_ξ with $t_\xi \perp s_{\xi+1}$. Then, the family $\{t_\xi \mid \xi < \eta\}$ is strongly incomparable.

Concerning strongly incomparable sets of nodes, we quote the following proposition whose proof is straightforward.

Proposition 2.1. *Suppose that $\{s_i \mid i \in I\}$ is a strongly incomparable set of nodes on the tree \mathcal{D} . Then the family $\{e_{s_i} \mid i \in I\}$ is equivalent to the usual basis of $c_0(I)$. More precisely, for any $n \in \mathbb{N}$, any $i_1, \dots, i_n \in I$ and any scalars a_1, \dots, a_n , we have*

$$\max_{1 \leq k \leq n} |a_k| \leq \left\| \sum_{k=1}^n a_k e_{s_{i_k}} \right\| \leq \sqrt{2} \max_{1 \leq k \leq n} |a_k|.$$

3. THE MAIN RESULTS

Suppose that $B = (a_\xi)_{\xi < \kappa} \in \Gamma$ is any branch. Then B can be naturally identified with a maximal segment of \mathcal{D} , namely $B = \{s_0 < s_1 < \dots < s_\eta < \dots\}$ where $s_0 = \emptyset$ and $s_\eta = (a_\xi)_{\xi < \eta}$ for any ordinal $\eta < \kappa$. In Section 2, we defined the linear functional $B^* : c_{00}(\mathcal{D}) \rightarrow \mathbb{R}$ by setting $B^*(x) = \sum_{s \in B} x(s)$. Clearly, $\|B^*\| = 1$. This functional can be extended to a bounded functional on X_κ , having the same norm and which is denoted again by B^* . Let also Γ^* denote the set which contains the functionals B^* defined above. Then Γ^* is a bounded subset of X_κ^* whose cardinality is equal to 2^κ .

This section is devoted to the study of the family Γ^* . Towards this direction, we first prove the following.

Theorem 3.1. *Suppose that $(B_n)_{n \in \mathbb{N}}$ is a sequence of branches such that $B_n \neq B_m$ for $n \neq m$. Then $(B_n^*)_{n \in \mathbb{N}}$ contains a subsequence equivalent to the usual ℓ_1 -basis.*

Proof. Consider the set \mathcal{A} consisting of all ordinals $\eta < \kappa$ which satisfy the following: there are nodes $\varphi \neq t$ with $\text{lev}(\varphi) = \text{lev}(t) = \eta$ and there are positive integers $m_1 \neq m_2$ such that $\varphi \in B_{m_1}$, $t \in B_{m_2}$. Clearly \mathcal{A} is a non-empty set, therefore we can consider its least element, say η . Then η can not be a limit ordinal. Indeed, let $\varphi = (a_\xi)_{\xi < \eta}$ and $t = (b_\xi)_{\xi < \eta}$ be as above. Since $\varphi \neq t$, there exists $\eta_1 < \eta$

with $a_{\eta_1} \neq b_{\eta_1}$. We set $\tilde{\varphi} = (a_\xi)_{\xi < \eta_1+1}$ and $\tilde{t} = (b_\xi)_{\xi < \eta_1+1}$. Now we observe that $\tilde{\varphi} \neq \tilde{t}$, these nodes are placed on the same level and $\tilde{\varphi} \leq \varphi$, $\tilde{t} \leq t$. Hence, $\tilde{\varphi} \in B_{m_1}$, $\tilde{t} \in B_{m_2}$. By the minimality of η , we conclude that $\eta = \eta_1 + 1$.

Furthermore, the minimality of η also implies that there exists a node s_1 on the level η_1 , so that $s_1 \in B_m$, for every $m \in \mathbb{N}$, and the nodes φ , t on the level $\eta = \eta_1 + 1$ are precisely the successors of s_1 . Now, we set $\varphi_1 = \varphi$ and $t_1 = t$. We may assume that there are infinitely many terms of the sequence $(B_m)_{m \in \mathbb{N}}$ which pass through the node φ_1 . Then we choose a branch B_{l_1} passing through the node t_1 (clearly such a branch does exist). B_{l_1} is just the first term of the desired subsequence.

We next set $N_1 = \{m \in \mathbb{N} \mid m > l_1 \text{ and } \varphi_1 \in B_m\}$. Then N_1 is an infinite subset of \mathbb{N} . Repeating the previous argument to the branches $(B_m)_{m \in N_1}$, we find an ordinal $\eta_2 > \eta_1 + 1$ and a node s_2 on the η_2 -th level with successors φ_2 and t_2 , such that

- all branches B_m , $m \in N_1$, pass through the node s_2 ;
- infinitely many branches of the sequence $(B_m)_{m \in N_1}$ pass through φ_2 and the set $\{m \in N_1 \mid t_2 \in B_m\}$ is non-empty.

We also choose a branch B_{l_2} so that $t_2 \in B_{l_2}$.

Continue in the obvious manner. We inductively construct a sequence $s_1 < s_2 < \dots$ of nodes of \mathcal{D} , with the successors of s_i denoted by φ_i and t_i , and a sequence $l_1 < l_2 < \dots$ of positive integers such that the following hold:

- (1) $s_1 < \varphi_1 \leq s_2 < \varphi_2 \leq s_3 \dots$;
- (2) $s_i \in B_{l_j}$ for any $j \geq i$, however the branches B_{l_j} , $j > i$, pass through the node φ_i while the branch B_{l_i} passes through the node t_i .

We prove now that the sequence $(B_{l_m}^*)_{m \in \mathbb{N}}$ is equivalent to the usual ℓ_1 -basis. Let $M \in \mathbb{N}$ and $a_1, \dots, a_M \in \mathbb{R}$ be given. We set $x = \sum_{i=1}^M \text{sgn}(a_i) e_{t_i}$. Condition (1) of the above construction implies that the sequence (t_i) is strongly incomparable. Hence by Proposition 2.1, we have $\|x\| = \sqrt{2}$. Furthermore, condition (2) implies that $t_i \in B_{l_i} \setminus \cup\{B_{l_j} \mid j \neq i\}$, thus $B_{l_j}(e_{t_i}) = \delta_{ij}$. Therefore:

$$\left\| \sum_{i=1}^M a_i B_{l_i}^* \right\| \geq \frac{1}{\|x\|} \left| \sum_{i=1}^M a_i B_{l_i}^*(x) \right| = \frac{1}{\sqrt{2}} \left| \sum_{i=1}^M a_i \text{sgn}(a_i) \right| = \frac{1}{\sqrt{2}} \sum_{i=1}^M |a_i|.$$

Clearly, we have $\left\| \sum_{i=1}^M a_i B_{l_i}^* \right\| \leq \sum_{i=1}^M |a_i|$ and the proof is complete. \square

Corollary 3.1. *The set Γ^* contains no weakly Cauchy sequence.*

We pass now to the second result concerning the set of functionals $\{B^* \mid B \in \Gamma\}$.

Theorem 3.2. *There exists no subset of Γ^* which is equivalent to the usual $\ell_1(\kappa^+)$ -basis.*

For the proof of the above theorem we need to establish some lemmas. Before proceeding, let us introduce some notation. First of all, if A is any set, then $|A|$ denotes the cardinality of A . Suppose now that $\Delta \subseteq \Gamma$ is a set of branches. For any node $s \in \mathcal{D}$, we denote Δ_s the set of all branches $B \in \Delta$ passing through s , that is $\Delta_s = \{B \in \Delta \mid s \in B\}$. We also set $\Delta_s^c = \Delta \setminus \Delta_s = \{B \in \Delta \mid s \notin B\}$.

Lemma 3.3. *Let $\Delta \subseteq \Gamma$ be a set of branches with $|\Delta| = \kappa^+$. Then there exists a node $s \in \mathcal{D}$ such that $|\Delta_{s \cup \{0\}}| = |\Delta_{s \cup \{1\}}| = \kappa^+$*

Proof. Assume that the assertion is not true. Then for every node $s \in \mathcal{D}$ there is a successor $s \cup \{\epsilon\}$ of s , where $\epsilon = 0$ or 1 , such that $|\Delta_{s \cup \{\epsilon\}}| < \kappa^+$. With this assumption and using transfinite induction we construct a branch $B = \{s_\eta\}_{\eta < \kappa} = \{s_0 < s_1 < \dots\}$ with the property that $|\Delta_{s_\eta}| = \kappa^+$ for any $\eta < \kappa$.

We start with $s_0 = \emptyset$. Clearly, $|\Delta_\emptyset| = |\Delta| = \kappa^+$. Suppose now that η is an ordinal, $\eta < \kappa$, and we have defined the nodes $\{s_\xi\}_{\xi < \eta}$ with $\text{lev}(s_\xi) = \xi$ and $|\Delta_{s_\xi}| = \kappa^+$ for any $\xi < \eta$.

If $\eta = \eta_0 + 1$, then by the inductive hypothesis we have $|\Delta_{s_{\eta_0}}| = \kappa^+$. Clearly, $\Delta_{s_{\eta_0}} = \Delta_{s_{\eta_0} \cup \{0\}} \cup \Delta_{s_{\eta_0} \cup \{1\}}$. Therefore, there exists a successor $s_{\eta_0} \cup \{\epsilon\}$ (where $\epsilon = 0$ or 1) of s_{η_0} such that $|\Delta_{s_{\eta_0} \cup \{\epsilon\}}| = \kappa^+$. Let $s_\eta = s_{\eta_0} \cup \{\epsilon\}$.

If η is a limit ordinal, we set $s_\eta = \bigcup_{\xi < \eta} s_\xi$. Then s_η is a node on the η -th level of \mathcal{D} . It remains to show that $|\Delta_{s_\eta}| = \kappa^+$. Since, $\Delta = \Delta_{s_\eta} \cup \Delta_{s_\eta}^c$, it suffices to prove that $|\Delta_{s_\eta}^c| \leq \kappa$.

Let us consider a branch B belonging to $\Delta_{s_\eta}^c$, that is $s_\eta \notin B$. We also denote S the initial segment $\{s_\xi\}_{\xi \leq \eta}$. We consider now the set \mathcal{A} containing all ordinals $\xi \leq \eta$ such that at the ξ -th level of \mathcal{D} , the segments B and S do not pass through the same node. The set \mathcal{A} is non-empty as $\eta \in \mathcal{A}$. Therefore \mathcal{A} has a minimum element, say ξ_0 . The minimality of ξ_0 implies that ξ_0 can not be a limit ordinal. Hence $\xi_0 = \xi + 1$. Further, it follows by the minimality of ξ_0 that at the level ξ , we have $s_\xi \in B$ and $s_\xi \in S$, while at the level $\xi + 1$, $s_{\xi+1} \in S$ and $s_{\xi+1} \notin B$. Consequently,

$$\begin{aligned} \Delta_{s_\eta}^c &= \bigcup_{\xi < \eta} \{B \in \Delta \mid s_\xi \in B \text{ and } s_{\xi+1} \notin B\} \\ &= \bigcup_{\xi < \eta} (\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c). \end{aligned}$$

Observe that $s_{\xi+1}$ is a successor of s_ξ , $|\Delta_{s_\xi}| = |\Delta_{s_{\xi+1}}| = \kappa^+$ and $\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c$ consists of all branches $B \in \Delta$ which pass through the other successor of s_ξ . By our assumption in the beginning of the proof, we have $|\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c| \leq \kappa$ and therefore $|\Delta_{s_\eta}^c| \leq \sum_{\xi < \eta} \kappa = \kappa$.

Therefore a branch $B = \{s_\eta\}_{\eta < \kappa}$ has been constructed with the property $|\Delta_{s_\eta}| = \kappa^+$ for any $\eta < \kappa$. To complete the proof of the lemma, we only need to repeat our last argument. Consider a branch $\tilde{B} \in \Delta$ with $\tilde{B} \neq B$. Let ξ_0 be the minimum ordinal such that at the ξ_0 -th level the branches \tilde{B}, B do not pass through the same node. The minimality of ξ_0 implies that $\xi_0 = \xi + 1$, $s_\xi \in \tilde{B}$ and $s_{\xi+1} \notin \tilde{B}$. Therefore

$$\Delta \subseteq \{B\} \cup \left(\bigcup_{\xi < \kappa} (\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c) \right).$$

Since $|\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c| \leq \kappa$, it follows that $|\Delta| \leq \kappa$ and we have reached a contradiction. \square

Lemma 3.4. *Let $\Delta \subset \Gamma$ be a set of branches with $|\Delta| = \kappa^+$. Then there exists a countable subtree T of \mathcal{D} , $T = \{t_1, t_2, t_3, \dots\}$, such that the following hold:*

- (1) $|\Delta_{t_m}| = \kappa^+$ for any node $t_m \in T$;
- (2) for any node $t_m \in T$ there exists a node $s_m \in \mathcal{D}$, so that $t_m \leq s_m$ and t_{2m}, t_{2m+1} are the successors of s_m (that is, when we look at the tree \mathcal{D} , then the successors of t_m still remain the successors of some node $s_m \in \mathcal{D}$).

Proof. Let $t_1 = \emptyset$. By Lemma 3.3, there exists a node $s_1 \in \mathcal{D}$, with $t_1 \leq s_1$ such that $|\Delta_{s_1 \cup \{0\}}| = |\Delta_{s_1 \cup \{1\}}| = \kappa^+$. We set $t_2 = s_1 \cup \{0\}$ and $t_3 = s_1 \cup \{1\}$. Then

t_2, t_3 are the successors of t_1 in T and they are the successors of s_1 when we look at the tree \mathcal{D} .

Applying Lemma 3.3 to the family $\Delta_{s_1 \cup \{0\}} = \Delta_{t_2}$ we find a node $s_2 \in \mathcal{D}$, with $t_2 \leq s_2$, such that $|\Delta_{s_2 \cup \{0\}}| = |\Delta_{s_2 \cup \{1\}}| = \kappa^+$. Then the successors of t_2 in T are the nodes $t_4 = s_2 \cup \{0\}$ and $t_5 = s_2 \cup \{1\}$. We continue in the obvious manner. \square

Proof of Theorem 3.2. Assume that $\Delta \subseteq \Gamma$ is a set of branches with $|\Delta| = \kappa^+$ and $\Delta^* = \{B^* \mid B \in \Delta\}$ is equivalent to the usual $\ell_1(\kappa^+)$ -basis. Then there exists a constant $\delta > 0$ such that for any $n \in \mathbb{N}$, any $B_1, \dots, B_n \in \Delta$ and any scalars a_1, \dots, a_n ,

$$\delta \sum_{i=1}^n |a_i| \leq \left\| \sum_{i=1}^n a_i B_i^* \right\| \leq \sum_{i=1}^n |a_i|.$$

Let T be the countable subtree of \mathcal{D} given by Lemma 3.4 and let $n \in \mathbb{N}$ be any positive integer. Then we choose branches B_1, \dots, B_n and B_{n+1}, \dots, B_{2n} belonging to Δ as follows. We work at the n -th level of T which consists of the nodes $t_{2^n}, t_{2^n+1}, t_{2^n+2}, \dots, t_{2^{n+1}-1}$. If we consider the pair t_{2^n}, t_{2^n+1} , the construction of the tree T implies that these nodes are the successors of some node of the tree \mathcal{D} . Therefore they belong to the same level of \mathcal{D} , say the level ξ_1 . Similarly the nodes t_{2^n+2}, t_{2^n+3} are placed on the same level of \mathcal{D} , say ξ_2 , and so on. Finally, let $\xi_{2^n-1} = \text{lev}(t_{2^{n+1}-2}) = \text{lev}(t_{2^{n+1}-1})$. We may assume, without loss of generality, that $\xi_1 = \max\{\xi_k \mid 1 \leq k \leq 2^n-1\}$. Then we choose branches B_1 and B_{n+1} of the family Δ such that B_1 passes through t_{2^n} and B_{n+1} passes through t_{2^n+1} (such branches exist by Lemma 3.4). If ψ_1 denotes the immediate predecessor (on the tree \mathcal{D}) of the nodes t_{2^n}, t_{2^n+1} , then the branches B_1, B_{n+1} coincide up to the level of ψ_1 and they separate each other at the next level.

The nodes t_{2^n}, t_{2^n+1} are followers of the node t_2 in the tree T . We now forget the followers of t_2 and we repeat the previous procedure to the nodes belonging to the n -th level of T which are followers of t_3 . That is, we detect the pair, say t_{2^n+2k}, t_{2^n+2k+1} , which is placed on the greatest level of \mathcal{D} (if this is not unique, we simply choose one). Then we choose branches B_2, B_{n+2} belonging to Δ such that B_2 passes through the left-hand node of the pair, i.e. the node t_{2^n+2k} , and B_{n+2} passes through the right-hand node t_{2^n+2k+1} . Let ψ_2 denote the immediate predecessor of t_{2^n+2k}, t_{2^n+2k+1} on the tree \mathcal{D} . Then $\text{lev}(\psi_1) \geq \text{lev}(\psi_2)$. The branches B_2, B_{n+2} coincide up to the level of ψ_2 . We also notice that the branches B_1, B_2 separate each other before the level of t_2, t_3 and this happens for the branches B_{n+1}, B_{n+2} . The nodes t_{2^n+2k}, t_{2^n+2k+1} are followers either of t_6 or t_7 . If t_6 is a predecessor of t_{2^n+2k}, t_{2^n+2k+1} , then we forget the followers of t_6 and we continue with the nodes belonging to the n -th level of T which are followers of t_7 .

After $n-1$ iterated applications of the previous argument, we find branches B_1, \dots, B_{n-1} and B_{n+1}, \dots, B_{2n-1} of the family Δ and nodes $\psi_1, \dots, \psi_{n-1}$ of \mathcal{D} . At this stage only one pair of nodes on the n -th level of T has been left. Let ψ_n be the immediate predecessor on \mathcal{D} of these nodes. We choose $B_n, B_{2n} \in \Delta$ such that B_n passes through the left-hand node and B_{2n} passes through the right-hand node.

Now we observe that the branches B_1, \dots, B_n are pairwise disjoint below the level of ψ_n and this is also true for the branches B_{n+1}, \dots, B_{2n} . Therefore, if $\eta_1 = \text{lev}(\psi_n)$ and $\eta_2 = \text{lev}(\psi_1)$, then the following hold.

- (1) All segments $B_i \cap \{s \mid \text{lev}(s) \geq \eta_2 + 1\}$, $i = 1, 2, \dots, 2n$, are pairwise disjoint.

- (2) The segments $B_i \cap \{s \mid \eta_1 + 1 \leq \text{lev}(s) \leq \eta_2\}$ for $i = 1, 2, \dots, n$ are pairwise disjoint. Hence they are admissible $(\eta_1 + 1)$ -($\eta_2 + 1$) segments. Similarly, $B_i \cap \{s \mid \eta_1 + 1 \leq \text{lev}(s) \leq \eta_2\}$, $i = n + 1, \dots, 2n$, form an admissible family.
- (3) $B_i \cap \{s \mid \text{lev}(s) \leq \eta_1\} = B_{n+i} \cap \{s \mid \text{lev}(s) \leq \eta_1\}$ for any $i = 1, 2, \dots, n$.
Let us also denote $S_i = B_i \cap \{s \mid \text{lev}(s) \leq \eta_1\}$.

After the choice of $(B_i)_{i=1}^{2n}$ has been completed, our next purpose is to estimate the norm of the functional $\sum_{i=1}^{2n} a_i B_i^*$ for any scalars a_1, \dots, a_{2n} and to contradict the assumption that Δ^* is equivalent to the usual $\ell_1(\kappa^+)$ -basis. For this reason, we consider a finitely supported vector $x = \sum_{s \in \mathcal{D}} \lambda_s e_s \in X_\kappa$ with $\|x\| \leq 1$. We can write $x = x_1 + x_2 + x_3$, where $x_1 = \sum_{\text{lev}(s) \leq \eta_1} \lambda_s e_s$, $x_2 = \sum_{\eta_1 + 1 \leq \text{lev}(s) \leq \eta_2} \lambda_s e_s$ and $x_3 = \sum_{\eta_2 + 1 \leq \text{lev}(s)} \lambda_s e_s$. Clearly, $\|x_j\| \leq \|x\| = 1$ for any $j = 1, 2, 3$. Then

$$\left| \sum_{i=1}^{2n} a_i B_i^*(x) \right| \leq \left| \sum_{i=1}^{2n} a_i B_i^*(x_1) \right| + \left| \sum_{i=1}^{2n} a_i B_i^*(x_2) \right| + \left| \sum_{i=1}^{2n} a_i B_i^*(x_3) \right|.$$

Now we have,

$$\begin{aligned} \left| \sum_{i=1}^{2n} a_i B_i^*(x_3) \right| &\leq \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{2n} |B_i^*(x_3)|^2 \right)^{1/2} \leq \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} \\ \left| \sum_{i=1}^{2n} a_i B_i^*(x_2) \right| &\leq \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^n |B_i^*(x_2)|^2 + \sum_{i=n+1}^{2n} |B_i^*(x_2)|^2 \right)^{1/2} \\ &\leq \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} (2\|x_2\|^2)^{1/2} \leq \sqrt{2} \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} \\ \left| \sum_{i=1}^{2n} a_i B_i^*(x_1) \right| &= \left| \sum_{i=1}^n (a_i B_i^*(x_1) + a_{n+i} B_{n+i}^*(x_1)) \right| \\ &= \left| \sum_{i=1}^n (a_i + a_{n+i}) S_i^*(x_1) \right| \leq \sum_{i=1}^n |a_i + a_{n+i}| |S_i^*(x_1)| \\ &\leq \sum_{i=1}^n |a_i + a_{n+i}|. \end{aligned}$$

Summarizing the above, for any finitely supported $x \in X_\kappa$ with $\|x\| \leq 1$ we have

$$\left| \sum_{i=1}^{2n} a_i B_i^*(x) \right| \leq (\sqrt{2} + 1) \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} + \sum_{i=1}^n |a_i + a_{n+i}|.$$

Therefore, $\|\sum_{i=1}^{2n} a_i B_i^*\| \leq (\sqrt{2} + 1) \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} + \sum_{i=1}^n |a_i + a_{n+i}|$. On the other hand, Δ^* is equivalent to the usual $\ell_1(\kappa^+)$ -basis. It follows that

$$\delta \sum_{i=1}^{2n} |a_i| \leq (\sqrt{2} + 1) \left(\sum_{i=1}^{2n} a_i^2 \right)^{1/2} + \sum_{i=1}^n |a_i + a_{n+i}|.$$

If we choose $a_1 = \dots = a_n = 1$ and $a_{n+1} = \dots = a_{2n} = -1$, then we obtain $\delta \leq \frac{\sqrt{2}+1}{\sqrt{2n}}$ for any $n \in \mathbb{N}$ and we reach a contradiction. \square

4. THE NON-SEPARABLE VERSION OF ROSENTHAL'S ℓ_1 -THEOREM

In this section, we show that we can not achieve a satisfactory extension of Rosenthal's ℓ_1 -theorem to spaces of the type $\ell_1(\kappa)$, for κ an uncountable cardinal. As it was mentioned in the introduction, this extension is possible in only one case, namely when both κ and $cf(\kappa)$ are strong limit cardinals. For the proof of this result we refer to [2] and we shall discuss the other cases.

Suppose first that κ is not a strong limit cardinal. This means that there exists a cardinal $\lambda < \kappa$ with $\kappa \leq 2^\lambda$. We now consider the space X_λ and the corresponding family of functionals $\Gamma^* \subset X_\lambda^*$. Then, Γ^* is a bounded subset of X_λ^* whose cardinality is equal to $2^\lambda \geq \kappa$. Further, by Corollary 3.1, the set Γ^* contains no weakly Cauchy sequence and, by Theorem 3.2, no subset of Γ^* is equivalent to the usual $\ell_1(\kappa)$ -basis.

We next consider the case where κ is strong limit but $cf(\kappa)$ is not a strong limit cardinal. This case is not so simple as the previous one, however it is essentially based on the arguments developed in Section 3.

Since $cf(\kappa)$ is not strong limit, there exists a cardinal $\lambda < cf(\kappa)$ with $cf(\kappa) \leq 2^\lambda$. By the definition of $cf(\kappa)$, there are cardinals $\{\kappa_i \mid i < cf(\kappa)\}$ such that $\kappa_i < \kappa$, for any ordinal $i < cf(\kappa)$, and $\kappa = \sum_{i < cf(\kappa)} \kappa_i$. We next consider the space X_κ and we choose a family of branches $A \subset \Gamma$ as follows. We focus on the level λ of the tree \mathcal{D} . This level consists of the nodes $\{0, 1\}^\lambda = \{(a_\xi)_{\xi < \lambda} \mid a_\xi = 0 \text{ or } 1\}$. Therefore, there are 2^λ nodes on the level λ . Since $cf(\kappa) \leq 2^\lambda$, we can choose nodes $\{t_i \mid i < cf(\kappa)\}$ on the level λ with $t_i \neq t_j$ provided that $i \neq j$. Now we observe that for any $i < cf(\kappa)$, the set of all branches passing through the node t_i has cardinality 2^κ . Hence, for any $i < cf(\kappa)$, we can choose a family of branches $A_i \subset \Gamma$ such that $|A_i| = \kappa_i$ and each branch belonging to A_i passes through the node t_i . Finally, let $A = \cup_{i < cf(\kappa)} A_i$ and let A^* be the family of the corresponding functionals, that is $A^* = \{B^* \mid B \in A\}$.

Clearly, the choice of the family A implies that $|A^*| = |A| = \sum_{i < cf(\kappa)} \kappa_i = \kappa$. Furthermore, by Corollary 3.1, A^* contains no weakly Cauchy sequence. So, it remains to show that no subset of A^* is equivalent to the usual $\ell_1(\kappa)$ -basis. The proof follows the lines of the proof of Theorem 3.2. We describe briefly the corresponding of Lemma 3.3.

Lemma 4.1. *Let Δ be a subset of A with $|\Delta| = \kappa$. Then there exists a node $s \in \mathcal{D}$ such that $lev(s) < \lambda$ and $|\Delta_{s \cup \{0\}}| = |\Delta_{s \cup \{1\}}| = \kappa$. (Recall that $\Delta_s = \{B \in \Delta \mid s \in B\}$.)*

Proof. Assuming that the assertion is not true, we construct an initial segment $S = \{s_\eta\}_{\eta < \lambda} = \{s_0 < s_1 < \dots\}$ such that $|\Delta_{s_\eta}| = \kappa$ for any $\eta < \lambda$. We start with $s_0 = \emptyset$. If $\eta = \eta_0 + 1$, then s_η is one of the followers of s_{η_0} . If η is a limit ordinal, then we set $s_\eta = \cup_{\xi < \eta} s_\xi$. Clearly, s_η is a node on the η -th level of \mathcal{D} . We next show that

$$\Delta_{s_\eta}^c = \cup_{\xi < \eta} (\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c).$$

Therefore, $|\Delta_{s_\eta}^c| = \sum_{\xi < \eta} |\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c| < \kappa$, since $|\Delta_{s_\xi} \cap \Delta_{s_{\xi+1}}^c| < \kappa$ and $\eta < \lambda < cf(\kappa)$. Hence $|\Delta_{s_\eta}| = \kappa$ and this completes the construction of S .

Finally, we set $s_\lambda = \cup_{\xi < \lambda} s_\xi$. Then s_λ belongs to the level λ and as previously we show $|\Delta_{s_\lambda}| = \kappa$. However, the choice of A indicates that $|\Delta_s| < \kappa$ for any node s on the level λ and we have reached a contradiction. \square

Using Lemma 4.1, we construct a countable subtree $T = \{t_1, t_2, t_3, \dots\}$ of \mathcal{D} such that:

- (1) $|\Delta_{t_m}| = \kappa$ for any $m = 1, 2, \dots$ (therefore, $\text{lev}(t_m) < \lambda$);
- (2) the successors t_{2m}, t_{2m+1} of the node t_m are the successors of some node $s_m \in \mathcal{D}$.

Finally, we repeat the proof of Theorem 3.2 to show that no subset Δ^* of A^* is equivalent to the usual $\ell_1(\kappa)$ -basis.

5. THE STRUCTURE OF THE SUBSPACES OF X_κ

The structure of the subspaces of the James Tree space (JT) and the Hagler Tree space (HT) has been studied extensively, since it has provided answers to several questions about Banach spaces. By analogy, the structure of the subspaces of X_κ seems quite interesting. This section is devoted to some remarks concerning this issue.

First of all, X_κ contains a lot of subspaces isomorphic to $c_0(\kappa)$. Indeed, let $B = \{s_\eta\}_{\eta < \kappa}$ be any branch and, for any $\eta < \kappa$, let t_η be the successor of s_η with $t_\eta \neq s_{\eta+1}$. Then $\{t_\eta \mid \eta < \kappa\}$ is a strongly incomparable family of nodes. By Proposition 2.1, it follows that the subspace $\overline{\text{span}}\{e_{t_\eta} \mid \eta < \kappa\}$ is isomorphic to $c_0(\kappa)$. Furthermore, it is easy to verify that for any ordinal $\eta < \kappa$ the subspace $\overline{\text{span}}\{e_s \mid s \in \{0, 1\}^\eta\}$ is isometrically isomorphic to the space $\ell_2(2^\eta)$. The main properties of the spaces JT and HT suggest now the following problem about the subspaces of X_κ .

Problem. Is it true that there exists no subspace of X_κ isomorphic to $\ell_1(\kappa)$?

Concerning the above problem, we prove a partial result. Assume that $B = \{s_\eta\}_{\eta < \kappa}$ is any branch of the tree \mathcal{D} . Then we show that the subspace generated by this branch, that is the subspace $\overline{\text{span}}\{e_{s_\eta}\}_{\eta < \kappa}$, does not contain any copy of $\ell_1(\kappa)$.

For our convenience, we first define a Banach space isometrically isomorphic to the subspace generated by any branch. Let κ be an infinite cardinal. We consider the vector space $c_{00}(\{\eta \mid \eta < \kappa\})$ consisting of all finitely supported functions $x : \{\eta \mid \eta < \kappa\} \rightarrow \mathbb{R}$. For any $x \in c_{00}(\{\eta \mid \eta < \kappa\})$, we set

$$\|x\| = \sup\{|S^*(x)|\}$$

where the supremum is taken over all segments $S \subseteq \{\eta \mid \eta < \kappa\}$. If E_κ denotes the completion of the normed space we have just defined, then E_κ is isometrically isomorphic to the subspace of X_κ generated by any branch.

As usual, for any ordinal $\eta < \kappa$, we consider the vector $e_\eta \in E_\kappa$ with $e_\eta(\xi) = 1$ if $\xi = \eta$ and $e_\eta(\xi) = 0$ otherwise. We now define some projections on the space E_κ . Let η be any ordinal, $\eta < \kappa$. We define $P_\eta : \text{span}\{e_\xi\}_{\xi < \kappa} \rightarrow \text{span}\{e_\xi\}_{\xi < \eta}$ as follows: if $x = \sum_{\xi < \kappa} x(\xi)e_\xi$ is finitely supported, then $P_\eta(x) = \sum_{\xi < \eta} x(\xi)e_\xi$. Clearly, P_η is a linear projection with $\|P_\eta\| = 1$. We can also extend P_η continuously and we obtain a projection $P_\eta : E_\kappa \rightarrow E_\kappa$ onto $\overline{\text{span}}\{e_\xi\}_{\xi < \eta}$ with $\|P_\eta\| = 1$. We next prove the following.

Proposition 5.1. *The space E_κ does not contain any isomorphic copy of $\ell_1(\kappa)$.*

Proof. Suppose, on the contrary, that $\ell_1(\kappa)$ embeds isomorphically into E_κ . Then we find a subset $\{x_\xi \mid \xi < \kappa\}$ of E_κ which is equivalent to the usual $\ell_1(\kappa)$ -basis.

Without loss of generality, we may assume that x_ξ is finitely supported and $\|x_\xi\| = 1$ for any $\xi < \kappa$.

We inductively construct a sequence $(y_m)_{m=0}^\infty$ belonging to $\text{span}\{e_\xi\}_{\xi < \kappa}$ with the following properties:

- (1) $\|y_m\| = 1$ for each m ;
- (2) if $A_m \subset \{\xi < \kappa\}$ is the support of y_m then $\max A_m < \min A_{m+1}$ for any m ;
- (3) $(y_m)_{m=0}^\infty$ is a block sequence of $(x_\xi)_{\xi < \kappa}$, that is there are ordinals $\eta_0 < \eta_1 < \dots$ so that $y_m \in \text{span}\{x_\xi \mid \eta_m \leq \xi < \eta_{m+1}\}$.

We start with $y_0 = x_0$, $\eta_0 = 0$ and $\eta_1 = 1$. Let $\xi_1 = \max A_0 + 1$. We claim that there exists $y \in \text{span}\{x_\xi\}_{\xi \geq 1}$, $y \neq 0$, such that $P_{\xi_1}(y) = 0$. Indeed, if we assume that $P_{\xi_1}(y) \neq 0$ for all $y \in \text{span}\{x_\xi\}_{\xi \geq 1}$, $y \neq 0$, then the linear operator $P_{\xi_1} : \text{span}\{x_\xi\}_{\xi \geq 1} \rightarrow \text{span}\{e_\xi\}_{\xi < \xi_1}$ is one-to-one. Since $\{x_\xi\}_{\xi \geq 1}$ are linearly independent, it follows that the (algebraic) dimension of the vector space $\text{span}\{e_\xi\}_{\xi < \xi_1}$ is equal to κ , which is a contradiction. Therefore, there is $y \in \text{span}\{x_\xi\}_{\xi \geq 1}$ such that $y \neq 0$ and $P_{\xi_1}(y) = 0$. We set $y_1 = y/\|y\|$. Since $P_{\xi_1}(y) = 0$, we have $\max A_0 < \min A_1$. Moreover, we can choose an ordinal $\eta_2 > \eta_1$ such that $y \in \text{span}\{x_\xi \mid \eta_1 \leq \xi < \eta_2\}$. Applying repeatedly the previous argument, we construct the desired sequence $(y_m)_{m=0}^\infty$.

Since $(x_\xi)_{\xi < \kappa}$ is equivalent to the usual $\ell_1(\kappa)$ -basis, it is easy to verify that the sequence (y_m) is equivalent to the usual ℓ_1 -basis. Furthermore, the sequence (y_m) belongs to $\text{span}\{e_\xi \mid \xi \in \cup_{m=0}^\infty A_m\}$. The latter space is isometrically isomorphic to E_{\aleph_0} , which in turn is isomorphic to c_0 (see [3]). That is, in a space isomorphic to c_0 we find a copy of ℓ_1 , which is a contradiction. \square

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